# Coding-Theoretic Methods for Sparse Recovery

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Abstract—We review connections between coding-theoretic objects and sparse learning problems. In particular, we show how seemingly different combinatorial objects such as error-correcting codes, combinatorial designs, spherical codes, compressed sensing matrices and group testing designs can be obtained from one another. The reductions enable one to translate upper and lower bounds on the parameters attainable by one object to another. We survey some of the well-known reductions in a unified presentation, and bring some existing gaps to attention. New reductions are also introduced; in particular, we bring up the notion of minimum L-wise distance of codes and show that this notion closely captures the combinatorial structure of RIP-2 matrices. Moreover, we show how this weaker variation of the minimum distance is related to combinatorial list-decoding properties of codes.

#### I. INTRODUCTION

Consider an n-dimensional vector  $x \in \mathbb{C}^N$  that is L-sparse, i.e., has L or less non-zero entries. The basic goal in compressed sensing is to design a *measurement matrix*  $M \in \mathbb{C}^{n \times N}$  such that from the measurement outcome

$$y := M \cdot x \in \mathbb{C}^n$$

it is information-theoretically possible to uniquely reconstruct x. Since x can be described by up to L complex numbers plus L integers in  $[N] := \{1, \ldots, N\}$  (that describe the support of the vector), it is natural to expect that the amount of measurements n can be made substantially less than the dimension N of the vector, even if one uses a set of linear forms as above to encode x. It turns out that the above intuition can be formalized and indeed there are measurement matrices with significantly smaller number of rows than columns [1], [2], [3], [4], [5]. In fact one can even obtain n = 2L by taking M to be a V and V are V and V and V and V and V and V are V and V and V and V and V are V and V and V are V and V and V are V and V and V are V and V and V and V are V and V and V and V are V and V are V and V and V are V and V and V are V and V are V and V and V are V and V and V are V and V are V and V are V and V a

Similar to compressed sensing, one can think of different sparse recovery problems with the goal of identifying objects that are known to have sparse representations. For example, compressed sensing can be extended to vectors over finite fields, which makes it essentially equivalent to the well-studied syndrome decoding problem of error-correcting codes, or to non-linear measurements. A particularly interesting class of non-linear measurements is characterized by disjunctions, which gives rise to a class of sparse recovery problems known as (non-adaptive) combinatorial group testing (cf. [7], [8]). In group testing, the measurement matrix and the sparse vector

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x both lie in the Boolean domain  $\{0,1\}$ . Then, the ith entry of the measurement y is defined as the logical expression

$$y(i) := (M_{i,1} \wedge x_1) \vee (M_{i,2} \wedge x_2) \vee \cdots \vee (M_{i,n} \wedge x_n),$$

where  $M_{i,j}$  denotes the *j*th entry of the *i*th row of M. Same as compressed sensing, group testing measurement matrices are known for  $n \ll N$ .

Even though we have defined the sparse recovery problems above in the most basic combinatorial form, in practice it is desirable to have measurement matrices with further qualities. For example, it is desirable to have an explicit construction of the measurement matrix; e.g., a polynomial-time algorithm for computing the entries of the matrix. Moreover, the *decoding algorithm* to infer the sparse vector from the measurement outcomes is of crucial importance and it is desirable to have as efficient a decoder as possible. Third, imprecisions are inevitable in practice and the design should be robust in presence of errors.

Going through the vast amount of literature in sparse recovery makes it evident that the theory of error-correcting codes proves to be of central importance in addressing the three basic requirements above. In this work, we revisit and highlight some of the known connections between coding theory and sparse recovery in a unified exposition, and moreover we introduce new connections. In particular, we study connections between coding-theoretic objects such as codes with large distance, list-decodable codes, combinatorial designs, and spherical codes to sparse recovery problems.

Coding theoretic methods have also been successfully applied to other sparse recovery problems, such as extensions of group testing to the threshold model and learning sparse hypergraphs, as well as low-rank matrix completion problems. However, due to the space limit, in this presentation we will only focus on the basic problems of compressed sensing and (noiseless) group testing. Moreover, we will only be able to emphasize on a few of the most basic reductions from coding-theoretic objects to measurement designs, and vice versa.

The rest of the paper is organized as follows. In Section I-A we review the notation that we use throughout the paper. Then, in Section II we introduce the notions of *Restricted Isometry Property* (RIP) and disjunct matrices that are central to compressed sensing and group testing, respectively. Section III shows how the minimum distance of error-correcting codes relate to the RIP. Section IV introduces the new idea of extending the notion of the minimum distance of codes to tuples of codewords, as opposed to pairs. Then, we

show a new result that this notion is more closely related to the RIP than the minimum distance. Section V shows the relationship between codes, combinatorial designs, and group testing schemes. Section VI touches upon some new connections between RIP matrices and list-decodable codes. Finally, Section VII concludes the work with possible future directions.

### A. Notation

For a vector  $v=(v_1,\ldots,v_n)$ , we use the convention  $v(i):=v_i$  for the ith entry of v and define  $\mathrm{supp}(v)\subseteq [n]$  to denote the support of v. For an  $n\times N$  matrix M, and a subset of column indices  $\mathcal{L}\subseteq [N]$ , the submatrix of M obtained by removing all columns of M outside  $\mathcal{L}$  is denoted by  $M|_{\mathcal{L}}$ . For a complex vector v, the  $\ell_p$  norm of v is denoted by  $\|v\|_p$ . When p=2, we may omit the subscript and simply write  $\|v\|$ . For a complex number  $a\in\mathcal{C}$ , the conjugate of a is denoted by  $a^*$ . For the most part in this write-up, we assume without loss of generality that q-ary codes are defined over the alphabet  $\mathbb{Z}_q$  even if we do not use the ring structure of  $\mathbb{Z}_q$ . For Boolean vectors x and y, we use  $\Delta(x,y)$  to denote the Hamming distance between x and y.

The statistical distance between two distributions  $\mathcal X$  and  $\mathcal Y$  with probability measures  $\Pr_{\mathcal X}(\cdot)$  and  $\Pr_{\mathcal Y}(\cdot)$  defined on the same finite space  $\Sigma$  is given by  $\frac{1}{2}\sum_{s\in\Sigma}|\Pr_{\mathcal X}(s)-\Pr_{\mathcal Y}(s)|$ , which is half the  $\ell_1$  distance of the two distributions when regarded as vectors of probabilities over  $\Sigma$ . Two distributions  $\mathcal X$  and  $\mathcal Y$  are said to be  $\epsilon$ -close if their statistical distance is at most  $\epsilon$ .

# II. COMBINATORICS OF SPARSE RECOVERY

It is easy to see that for the purpose of compressed sensing, a measurement matrix M can distinguish between all L-sparse vectors iff for every subset  $\mathcal L$  of up to 2L columns, the right kernel of the sub-matrix  $M|_{\mathcal L}$  is zero. This condition is in particular achieved by Vandermonde matrices [6]. However, in general such matrices need not be well-conditioned in the sense that the action of the matrix on sparse vectors may greatly affect their norm, which is not desirable in presence of imprecisions and/or noise in the measurements. A stronger condition would be to require each sub-matrix  $M|_{\mathcal L}$  to be nearly orthogonal. This gives rise to the notion of *Restricted Isometry Property* (RIP) as defined below.

**Definition 1.** Let  $p, \alpha > 0$  be real parameters. An  $n \times N$  matrix  $M \in \mathbb{C}^{n \times N}$  is said to satisfy RIP-p of order L with constant  $\alpha$  (or said to have L-RIP-p, in short) if for every  $\mathcal{L} \subseteq [N]$  with  $|\mathcal{L}| \leq L$  and every column vector  $x \in \mathbb{C}^{|\mathcal{L}|}$ , we have  $(1-\alpha)\|x\|_p \leq \|M|_{\mathcal{L}} \cdot x\|_p \leq (1+\alpha)\|x\|_p$ . The constant  $\alpha$  is sometimes omitted, in which case it is implicitly assumed to be an absolute constant in (0,1).

In this work, we will focus on the special case p=2. In this case, it is known that an RIP matrix is sufficient for distinguishing between sparse vectors even in presence of noise and when the vector being measured is *approximately sparse* (cf. [9], [10], [1]). Moreover, a linear program can be

used to reconstruct the sparse vector. Similar (but weaker) results are known about the RIP-1 (cf. [11]).

For group testing the following basic notion turns out to exactly capture the combinatorial structure needed for distinguishing between L-sparse vectors (cf. [7]):

**Definition 2.** An  $n \times N$  binary matrix is called L-disjunct if for any choice of L+1 columns  $M_0, \ldots, M_L$  of the matrix, we have  $\bigcup_{i \in [L]} \operatorname{supp}(M_i) \nsubseteq \operatorname{supp}(M_0)$ .

### III. FROM MINIMUM DISTANCE TO RIP

In this section we describe a few well known results about construction of RIP matrices from codes with good minimum distance properties. These techniques are used, for example, in [12], [13], [14] for deterministic construction of RIP matrices from specific families of codes. The reductions are based on the following simple embeddings of finite-domain vectors into the complex domain:

**Definition 3.** Let  $c \in \mathbb{Z}_q^n$  be a q-ary vector.

- 1) Let  $\zeta \in \mathbb{C}$  be a primitive qth root of unity. The spherical embedding of c, denoted by Sph(c), is a vector  $c' \in \mathbb{C}^n$  where for each  $i \in [n]$ , we define  $c'(i) := \zeta^{c(i)}/\sqrt{n}$ .
- 2) For any  $i \in \mathbb{Z}_q$ , denote by  $e_i$  the ith standard basis vector in  $\{0,1\}^q$ . That is,  $e_i(j) = 1$  if j = i+1 and  $e_i(j) = 0$  if  $j \neq i+1$ . The Boolean embedding of c, denoted by Bool(c), is a vector  $c'' \in \{0,1\}^{qn}$  obtained from c by replacing each element c(i) of c with the q-dimensional vector  $e_{c(i)}$ .

For example, consider the 4-dimensional binary vector  $c := (0, 1, 1, 0) \in \mathbb{F}_2^4$ . Then, we have  $\zeta = -1$  and

$$\begin{aligned} \mathsf{Sph}(c) &= (1, -1, -1, 1) \\ \mathsf{Bool}(c) &= (1, 0, 0, 1, 0, 1, 1, 0). \end{aligned}$$

The property that is later needed for the RIP constructions is the *bias* of the code, defined below.

**Definition 4.** A vector  $c \in \mathbb{Z}_q^n$  naturally induces a probability measure  $\mu_c$  on the alphabet  $\mathbb{Z}_q$ , where for each  $i \in \mathbb{Z}_q$ ,  $\mu_c(i)$  is the fraction of coordinate positions at which c is equal to i. The vector c is said to be  $\epsilon$ -biased if  $\mu_c$  is  $\epsilon$ -close to uniform.

**Definition 5.** A code  $C \subseteq \mathbb{Z}_q^n$  is said to be  $\epsilon$ -biased if, for every pair of distinct codewords  $c, c' \in C$ , the difference vector c-c' is  $\epsilon$ -biased.

Even though small bias is in general stronger than large minimum distance, for *balanced codes* as defined below the two notions are essentially equivalent, up to simple manipulations of the code.

**Definition 6.** A (possibly non-linear) code  $C \subseteq \mathbb{F}_q^n$  is called balanced if, for every  $c \in C$ , and every  $\alpha \in \mathbb{F}_q$ ,  $c + \alpha \mathbf{1} \in C$ , where  $\mathbf{1}$  denotes the all-ones vector.

**Definition 7.** Let  $C \subseteq \mathbb{Z}_q^n$  be a balanced code. Consider the equivalence relation between codewords that differ by a multiple of  $\mathbf{1} := (1, \dots, 1)$ . This partitions the codewords of C

into equivalence classes. Define C/1 to be any sub-code of C that picks exactly one codeword from each equivalence class.

**Proposition 8.** Let  $C \subseteq \mathbb{Z}_q^n$  be a balanced code with relative minimum distance at least  $1 - (1 + \epsilon)/q$ . Then the sub-code C/1 is  $\epsilon$ -biased.

*Proof:* Consider any pair of distinct codewords  $c,c' \in \mathcal{C}/1$  and define  $C' := \{c' + \alpha \mathbf{1} \colon \alpha \in \mathbb{Z}_q\}$ . Since  $\mathcal{C}$  is balanced,  $C' \subseteq \mathcal{C}$ . Moreover,  $c \notin C'$ , and therefore, the relative Hamming distance between c and any codeword in C' is at least  $1-(1+\epsilon)/q$ . In particular, the fraction of position at which c-c' is equal to any value  $\alpha \in \mathbb{Z}_q$  is at most  $(1+\epsilon)/q$  (since otherwise, the distinct vectors c and  $c'-\alpha \mathbf{1}$  would agree at more than  $(1+\epsilon)/q$  fraction of the positions, violating the minimum distance property). From the definition of statistical distance, we conclude that c-c' is  $\epsilon$ -biased.

Now we are ready to describe how small bias is related to geometric properties of the complex embeddings in Definition 3.

**Proposition 9.** Suppose  $c, c' \in \mathbb{Z}_q^n$  are so that c - c' is  $\epsilon$ -biased. Then  $|\langle \mathsf{Sph}(c), \mathsf{Sph}(c') \rangle| \leq 2\epsilon$ .

*Proof:* For  $i \in \mathbb{Z}_q$ , let  $p_i := |\{j \colon c(j) - c'(j) = i\}|/n$ . We know that the values  $p_i$  induce a probability distribution on  $\mathbb{Z}_q$  that is  $\epsilon$ -close to uniform. Define  $s_1 := \mathsf{Sph}(c)$  and  $s_2 := \mathsf{Sph}(c')$ . We have that

$$|\langle s, s' \rangle| = \left| \sum_{i \in [n]} s_1(i) s_2^*(i) \right|$$

$$= \left| \sum_{i \in [n]} \zeta^{c(i) - c'(i)} / n \right| = n \left| \sum_{i \in \mathbb{Z}_q} p_i \zeta^i / n \right|$$

$$= \left| \sum_{i \in \mathbb{Z}_q} (p_i - 1/q) \zeta^i \right|$$

$$\leq \sum_{i \in \mathbb{Z}_q} |p_i - 1/q| \leq 2\epsilon,$$
(1)

where (1) is due to the fact that  $\sum_{i \in \mathbb{Z}_a} \zeta^i = 0$ .

**Definition 10.** Let  $C \subseteq \mathbb{Z}_q^n$  be a code.

- 1) The spherical embedding of C is a complex  $n \times |C|$  matrix with columns indexed by the elements of C. The column corresponding to a codeword  $c \in C$  is Sph(c).
- 2) The Boolean embedding of C is a real  $n \times |C|$  matrix with 0/1 entries and columns indexed by the elements of C. The column corresponding to a codeword  $c \in C$  is Bool(c).

**Definition 11.** A set  $C \in \mathbb{C}^n$  is a spherical code<sup>1</sup> if each  $c \in C$  satisfies ||c|| = 1. Moreover, C is said to be  $\epsilon$ -coherent if, for any distinct  $c, c' \in C$ , we have  $|\langle c, c' \rangle| \leq \epsilon$ .

Using the above definition, Proposition 9 immediately implies the following.

**Corollary 12.** Let  $C \subseteq \mathbb{Z}_q^n$  be an  $\epsilon$ -biased code. Then the column set of Sph(C) forms a spherical code with coherence at most  $2\epsilon$ .

We are now ready to state how low-coherent spherical codes are related to RIP matrices. This is shown in the following well-known proposition:

**Proposition 13.** Suppose that the column set of an  $n \times N$  complex matrix M form an  $\epsilon$ -coherent spherical code. Then, M satisfies RIP-2 of order L with constant  $L\epsilon$ .

*Proof:* Consider an  $n \times L$  sub-matrix M' of M where  $M' := (M'_1 \mid \cdots \mid M'_L)$  and the  $M'_i$  are unit vectors in  $\mathbb{C}^n$ , and let  $x = (x_1, \ldots, x_L) \in \mathbb{C}^L$ . We can write

$$||M'x||^{2} - ||x||^{2} = \langle M'x, M'x \rangle - ||x||^{2}$$

$$= \langle \sum_{i \in [L]} x_{i} M'_{i}, \sum_{i \in [L]} x_{i} M'_{i} \rangle - ||x||^{2}$$

$$= \sum_{i \in [L]} x_{i}^{2} ||M_{i}||^{2} + \sum_{\substack{i,j \in [L] \\ i \neq j}} x_{i} x_{j} \langle M'_{i}, M'_{j} \rangle - ||x||^{2}$$

$$= \sum_{\substack{i,j \in [L] \\ i \neq j}} x_{i} x_{j} \langle M'_{i}, M'_{j} \rangle =: \eta.$$

And now we have

$$|\eta| \le \epsilon |\sum_{i,j} x_i x_j| \le \epsilon (\sum_{i \in [L]} x_i)^2 \le \epsilon ||x||_1^2 \le L\epsilon ||x||_2^2,$$

where the last inequality is by Cauchy-Schwarz.

The above proposition can be combined with Proposition 8 and Corollary 12 to show the following result.

**Corollary 14.** Let  $C \subseteq \mathbb{Z}_q^n$  be a balanced code with relative minimum distance at least  $1 - (1 + \epsilon)/q$ . Then, Sph(C/1) satisfies RIP-2 of order L with constant  $2L\epsilon$ .

As for Boolean embedding  $Bool(\cdot)$ , the following observation is easy to verify:

**Proposition 15.** Let  $C \subseteq \mathbb{Z}_q^n$  be a code with relative minimum distance at least  $\delta$ . Then, columns of  $\mathsf{Bool}(C)/\sqrt{n}$  form a  $(1-\delta)$ -coherent spherical code.

Combined with Proposition 13, we see that Boolean embedding can also result in RIP matrices.

**Corollary 16.** Let  $C \subseteq \mathbb{Z}_q^n$  be a code with relative minimum distance at least  $1 - (1 + \epsilon)/q$ . Then,  $\mathsf{Bool}(C)/\sqrt{n}$  satisfies RIP-2 of order L with constant  $(1 + \epsilon)L/q$ .

Now we consider instantiations of the above result with asymptotically good families of codes. Various positive and negative bounds are known for rate-distance trade-offs achievable by error-correcting codes. On the positive side, the *Gilbert-Varshamov bound* on codes [15], [16] states that for every alphabet size q>1 and constant  $\delta\in[0,1-1/q)$ , there are q-qry codes with rate

$$R \ge 1 - h_a(\delta) - o(1),\tag{2}$$

<sup>&</sup>lt;sup>1</sup>Traditionally spherical codes are defined under the constraint of having upper bounded (but possibly negative) mutual inner products. In this work we will require them to have low *coherence*, which is a stronger property.

where  $h_q(\cdot)$  is the q-ary entropy function defined as

$$h_q(\delta) := \delta \log_q(q-1) - \delta \log_q(\delta) - (1-\delta) \log_q(1-\delta).$$

This bound is achieved by a random linear code (assuming a prime power alphabet size) with overwhelming probability, and one can also make sure that the code is balanced, by forcing the all-ones word to be in the code. When  $\delta=1-(1+\epsilon)/q$ , the bound  $1-h_q(\delta)$  becomes

$$\frac{\epsilon^2}{2(q-1)\ln q} + \frac{\epsilon^3(q-2)}{6(q-1)^2\ln q} + O_q(\epsilon^4) = \Omega(\epsilon^2/(q\log q)).$$
(3

Now let us instantiate the above results with a balanced q-ary code  $\mathcal{C} \subseteq \mathbb{Z}_q^n$  on the Gilbert-Varshamov bound and with relative minimum distance  $1-(1+\epsilon)/q$ . First, consider the spherical encoding  $\operatorname{Sph}(\mathcal{C}/1)$  and suppose that we wish to obtain an  $n \times N$  RIP-2 matrix of order L with a fixed constant  $\alpha$ . In order to apply Corollary 14, we need to set  $\epsilon = \alpha/(2L)$ . In this case, the Gilbert-Varshamov bound implies that the rate R of  $\mathcal{C}$  can be at least  $\Omega(\epsilon^2/(q\log q)) = \Omega(\alpha^2/(L^2q\log q))$ . The number of columns of the resulting matrix is  $N=q^{Rn-1}$ . Therefore, we have

$$\log N = (Rn - 1)\log q = \Omega(\alpha 2n/(L^2q)),$$

or in other words,

$$n = O(L^2(\log N)q/\alpha^2) = O_{\alpha,q}(L^2\log N). \tag{4}$$

We remark that Porat and Rothschild [17] show how to derandomize the probabilistic construction of linear codes on the Gilbert-Varshamov bound for any fixed prime power alphabet q. They design a deterministic algorithm for constructing the generator matrix of the code in time  $O(nq^{Rn})$ , where R is the rate<sup>2</sup>. This running time is in nearly *linear* in the number of the entries of the resulting RIP matrix.

It is well known that there are RIP-2 matrices of order L with  $n=O(L\log(N/L))$  rows and this bound is achieved by several probabilistic constructions (in particular, independent Bernoulli  $\pm 1/\sqrt{n}$  entries) [18], [19]. However we see that even using codes on the Gilbert-Varshamov bound the number of rows of the RIP matrix obtained from Corollary 14 becomes larger by a multiplicative factor of about  $\Omega(L)$ . To see whether this can be improved, we consider negative bounds on the rate-distance trade-offs of codes.

For our range of parameters, the best known negative bounds on the rate-distance of error-correcting codes (that show upper bounds on the rate of any code with a certain minimum distance) are given by linear-programming techniques. In particular, the linear programming bound due to McEliece, Rodemich, Rumsey, and Welch (cf. [20, Chapter 5]) states that, asymptotically, any binary code with relative minimum distance at least  $\delta$  and rate R must satisfy

$$R \le h(1/2 - \sqrt{\delta(1-\delta)}) + o(1).$$

This bound can be generalized to q-ary codes as follows (see [21]).

$$R \le h_q \left( \frac{1}{q} (q - 1 - (q - 2)\delta - 2\sqrt{(q - 1)\delta(1 - \delta)}) \right) + o(1).$$
(5)

For any fixed q, and for  $\delta = 1 - (1+\epsilon)/q$ , this bound simplifies to  $R = O(\epsilon^2 \log(1/\epsilon))$ . Using simple calculations as before, we conclude that the RIP-2 matrix construction of Corollary 14 always requires  $n = \Omega(L^2(\log N)/\log L)$  rows, regardless of the code being used.

The RIP matrices constructed from Corollary 14 require a factor  $\tilde{\Omega}(L^2)$  in the number of rows due to the fact that their column set forms a spherical code. It is known that any  $\epsilon$ -coherent spherical code of size N over  $\mathbb{C}^n$  must satisfy the following (cf. [22])

$$\epsilon^2 = \Omega\left(\frac{\log N}{n\log(n/\log N)}\right),\tag{6}$$

which implies  $n = \Omega((\log N)/(\epsilon^2 \log(1/\epsilon)))$ . Therefore, the factor  $\epsilon^2$  in the denominator of the bound on n (which translates to a factor  $L^2$  in the RIP setting) is necessary.

On the positive side, the reduction above from the codes on the Gilbert-Varshamov bound indirectly shows that spherical codes with coherence  $\epsilon = O((\log N)/n)$  (i.e.,  $n = O(\epsilon^2 \log N)$ ) exist and can be attained using probabilistic constructions. On the negative side, the lower bound (6) can be translated (using the reduction from error-correcting codes to spherical codes) to upper bounds on the attainable rates of q-ary codes with distance close to 1 - 1/q. This results in an indirect upper bound comparable to what the linear programming bound (5) implies.

Now we turn to the construction of RIP matrices from the Boolean embedding of error-correcting codes obtained in Corollary 16. In order to obtain an RIP-2 matrix of order L with constant  $\alpha$ , by Corollary 16 it suffices to have a code  $\mathcal{C}\subseteq\mathbb{Z}_q^n$  attaining the Gilbert-Varshamov bound with relative minimum distance at least  $1-(1+\epsilon)/q$  and  $\epsilon \leq (\alpha q/L)-1$ . For a fixed constant  $\alpha$ , we can set q=O(L) large enough (e.g.,  $q=2L/\alpha$ ) and choose  $\epsilon$  to be a small absolute constant (e.g.,  $\epsilon=.01$ ) so that the above condition is satisfied. The resulting matrix would have  $N:=|\mathcal{C}|$  columns and n':=nq rows, with entries that are either 0 or  $1/\sqrt{n}$ . Moreover, the matrix is rather sparse in that all but a 1/q fraction of the entries are zeros.

Now, the Gilbert-Varshamov bound (2) implies that the rate R of  $\mathcal C$  can be made at least  $\Omega(\epsilon^2/(q\log q)) = \Omega(1/(q\log q))$ . Thus we have

$$\log N = \log |\mathcal{C}| = (Rn'/q) \log q = \Omega(n'/q^2)$$

which gives  $n' = O(q^2 \log N) = O(L^2 \log N)$ . This is comparable to the bound (4) that we obtained from spherical embedding of codes. Similar to the case of spherical codes, Boolean embedding allows us to translate positive bounds on the rate-distance trade-off of codes (e.g., the Gilbert-Varshamov bound) to upper bounds on the coherence of

<sup>&</sup>lt;sup>2</sup> The algorithm can be adapted to ensure that the obtained code is balanced.

spherical codes as well as upper bounds on the number of rows of RIP-2 matrices. Conversely, through Boolean embedding, lower bounds on the coherence of spherical codes and lower bounds on the number of rows of RIP-2 matrices translate into impossibility bounds on the rate-distance trade-off of error-correcting codes, the former being comparable to the linear programming bound (5) when the relative minimum distance is around 1-1/q, but the latter is much weaker (namely, comparable to the Plotkin bound on codes [20, Chapter 5] which is, over small alphabets, much weaker than the linear programming bounds).

#### IV. FROM AVERAGE DISTANCE TO RIP

As we saw in the previous section, the quadratic dependence on sparsity L is unavoidable when the column set of an RIP matrix forms a low-coherence spherical code. In this section we introduce the notion of L-wise distance that turns out to be more closely related to the RIP.

**Definition 17.** Let  $c_1, \ldots, c_L \in \mathbb{Z}_q^n$  be L vectors. The average distance of  $c_1, \ldots, c_L$  is defined in the natural way

$$\operatorname{dist}_L(c_1,\ldots,c_L) = rac{1}{n{L\choose 2}} \left\{ \sum_{1 \leq i < j \leq L} \Delta(c_i,c_j) \right\},$$

where  $\Delta(c_i, c_j)$  is the Hamming distance between  $c_i$  and  $c_j$ .

**Definition 18.** Let  $C \subseteq \mathbb{Z}_q^n$  be a code, and L be an integer where  $1 < L \le |C|$ . Define the L-wise distance of C as

$$\mathsf{dist}_L(\mathcal{C}) := \min_{\{c_1, ..., c_L\} \subseteq \mathcal{C}} \mathsf{dist}_L(c_1, \dots, c_L).$$

The special case L=2 is equal to the minimum relative distance of the code. For the other extreme case,  $L=|\mathcal{C}|$ , the L-wise distance of the code is the average relative distance over all codeword pairs. For linear codes, this quantity is the expected relative weight of a random codeword, given by

$$\mathsf{dist}_{|\mathcal{C}|}(\mathcal{C}) = \frac{(q-1)|\{i \in [n] \colon (\exists (c_1, \dots, c_n) \in \mathcal{C}), c_i \neq 0\}|}{qn}.$$

Thus, as long as the code is non-constant at all positions, its  $|\mathcal{C}|$ -wise distance is equal to (1-1/q). Also, a simple exercise shows the "monotonicity property" that for any code  $\mathcal{C}$ , and  $L' \geq L$ ,  $\operatorname{dist}_{L'}(\mathcal{C}) \geq \operatorname{dist}_L(\mathcal{C})$ .

We will use the notion of *flat RIP* below from [23].

**Definition 19.** Let  $\alpha > 0$  be a real parameter. An  $n \times N$  matrix  $M \in \mathbb{C}^{n \times N}$  with columns  $M_1, \ldots M_N \in \mathbb{C}^n$  is said to satisfy flat RIP of order L with constant  $\alpha$  if for all  $i \in [N]$ ,  $||M_i|| = 1$  and moreover, for any disjoint  $\mathcal{L}_1, \mathcal{L}_2 \subseteq [N]$  with  $|\mathcal{L}_1| = |\mathcal{L}_2| \leq L$  we have

$$\left| \left\langle \sum_{i \in \mathcal{L}_1} M_i, \sum_{i \in \mathcal{L}_2} M_i \right\rangle \right| \le \alpha \sqrt{|\mathcal{L}_1| |\mathcal{L}_2|} = \alpha |\mathcal{L}_1|.$$

The original definition of flat RIP in [23] is stronger and does not assume the two sets  $|\mathcal{L}_1|$  and  $|\mathcal{L}_2|$  have equal sizes. However, adding the extra constraint does not affect the result that we use from their work (Lemma 21 below).

A straightforward exercise shows that the standard RIP-2 is no weaker than the flat RIP, namely,

**Proposition 20.** Suppose a matrix M satisfies RIP-2 of order 2L with constant  $\alpha$ . Then, M satisfies flat RIP of order L with constant  $O(\alpha)$ .

More interestingly, the two notions turn out to be essentially *equivalent* (up to a logarithmic loss in the RIP constant) in light of the following result by Bourgain et al.:

**Lemma 21.** [23] Let  $L \ge 2^{10}$  and suppose that a matrix M satisfies flat RIP of order L with constant  $\alpha$ . Then M satisfies RIP-2 of order 2L with constant  $44\alpha \log L$ .

The notion of L-wise distance is a relaxed variation of the minimum distance, where the distance is averaged over various choices of L distinct codewords, as opposed to only two. Similarly, the notions of  $\epsilon$ -biased codes and spherical codes can be relaxed to L-wise forms and one can obtain various generalizations of the results presented in Section III to codes satisfying the relaxed notion of L-wise distance.

For clarity of presentation, for the remainder of this section we only focus on binary codes. In this case, if the code  $\mathcal C$  with L-wise distance at least  $1/2-\epsilon$  contains the all-ones word, one can simply show that not only the average distance of any choice of L codewords in  $\mathcal C/1$  is at least  $1/2-\epsilon$ , but this quantity is also no more than  $1/2+\epsilon$  (to see this, it suffices to note that the average distance of L codewords plus the average distance of their negations equals one). Let us call codes satisfying this stronger property L-wise  $\epsilon$ -biased:

**Definition 22.** Let  $C \subseteq \mathbb{Z}_2^n$  be a code, and L be an integer where  $1 < L \le |C|$ . Then, C is called L-wise  $\epsilon$ -biased if

$$\max_{\{c_1,\ldots,c_L\}\subseteq\mathcal{C}}\left|\mathsf{dist}_L(c_1,\ldots,c_L)-1/2\right|\leq\epsilon.$$

The result below shows how the flat RIP and L-wise distance are related. Again, the result is only presented for binary codes and the extension to q-ary codes is straightforward.

**Lemma 23.** Suppose  $C \subseteq \mathbb{Z}_2^n$  is such that, for a positive integer  $L_0$  and all  $L \leq 2L_0$ , C is L-wise  $(\alpha/L)$ -biased. Then, Sph(C) satisfies flat RIP of order  $L_0$  with constant  $4\alpha$ .

*Proof:* Fix any  $L' \leq 2L_0$  and any collection  $c_1, \ldots, c_{L'}$  of the codewords in C. Define

$$\eta(c_1, \dots, c_{L'}) := \sum_{1 \leq i < j \leq L'} \langle \mathsf{Sph}(c_i), \mathsf{Sph}(c_j) \rangle$$

$$= \sum_{1 \leq i < j \leq L'} (2\Delta(c_i, c_j)/n - 1)$$

$$\in \left[ -2\frac{\alpha}{L'} \binom{L'}{2}, +2\frac{\alpha}{L'} \binom{L'}{2} \right] \qquad (7)$$

$$\in \left[ -\alpha L', +\alpha L' \right], \qquad (8)$$

where (7) is due to the small-bias assumption on C.

Now, let  $L \leq L_0$  and  $M_1, \ldots, M_{2L}$  be distinct columns of  $Sph(\mathcal{C}/\mathbf{1})$  corresponding to distinct codewords  $c'_1, \ldots, c'_{2L}$  in

C. Now, from Definition 19, we need to bound the quantity

$$\begin{split} \eta' &:= &\left\langle \sum_{i \in [L]} M_i, \sum_{L < i \leq 2L} M_i \right\rangle \\ &= &\left. \sum_{1 \leq i < j \leq 2L} \langle M_i, M_j \rangle - \sum_{1 \leq i < j \leq L} \langle M_i, M_j \rangle - \right. \\ &\left. \sum_{L+1 \leq i < j \leq 2L} \langle M_i, M_j \rangle. \end{split}$$

Now, the absolute value of  $\eta'$  can be bounded as

$$|\eta'| \leq |\eta(c_1,\ldots,c_{2L})| + |\eta(c_1,\ldots,c_L)| + |\eta(c_{L+1},\ldots,c_{2L})| \stackrel{(\star)}{\leq} 4\alpha L,$$

where  $(\star)$  is from (8).

Note that, contrary to Corollary 14, the above result does not require the code to have an extremal minimum distance. In principle, C can have a minimum distance bounded away from 1/2 by a constant (depending on the constant  $\alpha$ ) and still satisfy the conditions of Lemma 23.

The above result is also valid in the reverse direction, as follows.

**Lemma 24.** Let M be an  $n \times N$  matrix with entries in  $\{-1/\sqrt{n}, +1/\sqrt{n}\}$  satisfying the flat RIP of order  $L_0$  with constant  $\alpha$ . Then, columns of M form the spherical encoding of a code  $\mathcal{C} \subseteq \mathbb{Z}_2^n$  such that for any  $L \leq L_0$ , the code  $\mathcal{C}$  is L-wise  $O(\alpha/L)$ -biased.

*Proof:* Assume L is even (the odd case is similar). Consider any L distinct columns  $M_1,\ldots,M_L$  of M and observe that

$$\eta := \sum_{\substack{1 \leq i < j \leq [L] \\ 1 \leq i \leq L}} \langle M_i, M_j \rangle$$

$$= \sum_{\substack{\mathcal{L} \subseteq [L] \\ |\mathcal{L}| - L \\ j \in [L] \setminus \mathcal{L}}} \langle M_i, M_j \rangle / \binom{L}{\frac{L}{2} - 1}.$$

By the flat RIP, each term  $\sum_{i\in\mathcal{L}}\sum_{j\in[L]\setminus\mathcal{L}}\langle M_i,M_j\rangle$  is upper bounded in absolute value by  $\alpha L/2$ , and therefore, the above equation simplifies in absolute value to  $|\eta|=O(\alpha L)$ . Now suppose the codewords corresponding to  $M_1,\ldots,M_L$  are  $c_1,\ldots,c_L$ . The L-wise distance of these codewords can be written as

$$\operatorname{dist}_{L}(c_{1},\ldots,c_{L}) = \frac{1}{\binom{L}{2}} \left( \sum_{1 \leq i < j \leq L} \frac{1 + \langle M_{i}, M_{j} \rangle}{2} \right)$$
$$= \frac{1}{2} + \eta / \binom{L}{2}.$$

Hence, 
$$|\operatorname{dist}_L(c_1,\ldots,c_L)-1/2|=|\eta|/{L\choose 2}=O(\alpha/L).$$

### V. DESIGNS AND DISJUNCT MATRICES

In this section we turn to the problem of combinatorial group testing, and in particular discuss coding-theoretic constructions of disjunct matrices. One of the foremost constructions dates back to the work of Kautz and Singleton [24],

who used Reed-Solomon codes for the purpose of constructing disjunct matrices<sup>3</sup>. This work results in a general framework for construction of disjunct matrices through *combinatorial designs*, which are defined as follows.

**Definition 25.** An (n, n', r)-design is a set system  $S_1, \ldots, S_N \subseteq [n]$  such that the size of each set is n' and for every pair  $i, j \in [N]$   $(i \neq j)$  we have  $|S_i \cap S_j| \leq r$ .

The following simple observations show that designs can be used to construct disjunct matrices, and can in turn be obtained from error-correcting codes:

**Lemma 26.** Let  $\mathcal{D} = \{S_1, \ldots, S_N\}$  be an (n, n', r)-design, and consider the binary  $n \times N$  matrix M induced by  $\mathcal{D}$  where the ith column of M is supported on  $S_i$ . Then, M is L-disjunct provided that Lr < n'.

*Proof:* It suffices to observe that in Definition 2, each of the  $M_i$  for  $i \in [L]$  contains at most r of the n' elements on  $supp(M_0)$ .

**Lemma 27.** Let  $C = \{c_1, \ldots, c_N\} \subseteq \mathbb{Z}_q^{n'}$  be a code with minimum Hamming distance at least d. For n := n'q, consider the set system  $\mathcal{D} := \{S_i : i \in [N]\}$  defined from the Boolean embedding of C as follows:  $S_i := \text{supp}(\mathsf{Bool}(c_i))$ . Then,  $\mathcal{D}$  is an (n, n', n' - d)-design.

*Proof:* Observe that intersection size  $|S_i \cap S_j|$ , for  $i \neq j$ , is equal to  $n' - \Delta(c_i, c_j) \leq n' - d$ . The rest of the conditions are trivial.

Now let us instantiate the above lemmas with a k-dimensional Reed-Solomon code, as in [24]. In this case, the alphabet size q can be made equal to the block length n' (assuming that n' is a prime power). From Lemma 27, the resulting (n,n',r)-design satisfies  $n=n'^2$ , r=n'-(n'-k)=k (since the minimum distance of the code is n'-k+1), and  $\log N=k\log q=r\log n'>r$ . Furthermore, by Lemma 26, characteristic vectors of the resulting design form a disjunct matrix with sparsity parameter  $L\approx n'/r$ . Therefore, the number of rows n can be upper bounded as  $n=n'^2\approx (rL)^2<(L\log N)^2$ .

As a second example, consider choosing a q-ary code on the Gilbert-Varshamov bound with minimum Hamming distance at least  $d:=n'-(1+\epsilon)n'/q$ , for some small (and fixed) constant  $\epsilon>0$ . Recall that the rate R of the code satisfies  $R=\Omega(\epsilon^2/(q\log q))$ . This time, we obtain an (n,n',r)-design with  $r=n'-d=(1+\epsilon)n'/q$ ,  $n=n'q=(1+\epsilon)n'^2/r=O(n'^2/r)$  and  $\log N=Rn'\log q=\Omega(\epsilon^2/q)=\Omega(r\epsilon^2/(1+\epsilon))=\Omega(r)$ . Now lemma 26 implies that the measurement matrix that has the Boolean embedding of the codewords as its columns is L-disjunct for  $L\approx n'/r$ . Note that since  $q=(1+\epsilon)n'/r$ , we must choose  $q=\Omega(L)$  for the bounds to follow. Altogether, we obtain  $n=n'q=O(n'^2/r)=O(L^2\log N)$ .

Probabilistic arguments can be used to show that (n, n', r)-designs of size N exist for  $n = O(n'^2N^{1/r}/r)$ , and moreover,

<sup>&</sup>lt;sup>3</sup>The work of Kautz and Singleton aims to construct superimposed codes, which are closely related to disjunct matrices.

this bound is known to be nearly tight (cf. [25] and [7, Ch. 7]). Therefore, we see that the design obtained from codes on the Gilbert-Varshamov bounds for which  $nr/n'^2 = O(1)$  and  $\log N = \Omega(r)$  essentially achieves the best possible bounds.

Regarding the existence of disjunct matrices, it is known that L-disjunct matrices exists with  $n = O(L^2 \log N)$  rows (using the probabilistic method) and moreover, any L-disjunct matrix must satisfy  $n = \Omega(L^2 \log_L N)$  (cf. [7, Ch. 7]). Again, we see that the disjunct matrices obtained from codes on the Gilbert-Varshamov bounds are essentially optimal. Moreover, such matrices can be generated in polynomial time in the size of the matrix using the result of Porat and Rothschild [17].

## VI. LIST DECODING AND SPARSE RECOVERY

As we saw in Section IV, the relaxed notion of L-wise distance essentially captures the RIP-2 for matrices with  $\pm 1/\sqrt{n}$  entries. In this section, we relate this notion to the standard notion of combinatorial list-decoding that has been extensively studied in the coding-theory literature.

We remark that the notion of soft-decision list-decodable codes has been used for construction of RIP-1 matrices, and it is known that optimal RIP-1 matrices can be constructed from optimal *soft-decision* list-decodable codes which, in particular, imply optimal unbalanced lossless expander graphs (see [26], [27], [11] and the references therein for the construction of RIP-1 matrices from expander graphs and [28] for the reduction from codes to expander graphs). The goal is this section is to show how list-decoding is related to the more geometric property RIP-2.

**Definition 28.** A code  $C \subseteq \mathbb{Z}_q^n$  is  $(L, \rho)$ -list decodable if for any  $x \in \mathbb{Z}_q^n$ , we have  $|\mathcal{B}(x, \rho) \cap C| < L$ , where  $\mathcal{B}(x, \rho)$  denotes the Hamming ball of radius  $\rho n$  around x.

In the following lemma, we show that codes with good L-wise distance have good list-decoding properties.

**Lemma 29.** Suppose that the L-wise distance of a code  $C \subseteq \mathbb{Z}_2^n$  is at least  $1/2 - \epsilon^2$ , where  $L = O(1/\epsilon^2)$ . Then, C is  $(O(1/\epsilon^2), 1/2 - \epsilon)$ -list decodable.

*Proof:* The proof idea is inspired by a geometric proof of the Johnson's bound due to Guruswami and Sudan [29]. By the end of the proof, we will determine an  $L' = O(1/\epsilon^2)$  satisfying  $L' \geq L$  such that the assumption that  $\mathcal C$  is not  $(L',\epsilon)$ -list decodable leads to a contradiction.

Now, for the sake of contradiction, consider any  $x \in \mathbb{Z}_2^n$  for which  $\mathcal{C} \cap \mathcal{B}(x, 1/2 - \epsilon)$  has size at least L'. Take any set of distinct codewords

$$c_1,\ldots,c_{L'}\in\mathcal{C}\cap\mathcal{B}(x,1/2-\epsilon)$$

and consider the spherical encodings  $v_0 := \mathsf{Sph}(x), \ v_1 := \mathsf{Sph}(c_1), \ldots, v_\ell := \mathsf{Sph}(c_{L'})$ . By the monotonicity property of the L-wise distance, we know that  $\mathsf{dist}_{L'}(c_1, \ldots, c_{L'}) \ge 1/2 - \epsilon^2$ . For spherical embeddings, this translates to

$$\sum_{1 \le i < j \le L'} \langle v_i, v_j \rangle = \binom{L'}{2} (1 - 2 \mathsf{dist}_{L'}(c_1, \dots, c_{L'})) \le 2L'^2 \epsilon^2$$

Also, since the relative Hamming distance between x and any  $c_i$  is at most  $1/2 - \epsilon$ , we get

$$(\forall i \in [L']) \quad \langle v_i, v_0 \rangle = (1 - 2\Delta(c_i, x)/n) \ge 2\epsilon. \tag{10}$$

Using (10), for every  $i \in [L']$  and parameter  $\beta > 0$ ,

$$\langle v_i - \beta v_0, v_i - \beta v_0 \rangle = 1 + \beta^2 - 2\beta \langle v_i, v_0 \rangle \le 1 + \beta^2 - 4\epsilon\beta. \tag{11}$$

Similarly, for  $1 \le i < j \le L'$  we can write

$$\langle v_i - \beta v_0, v_j - \beta v_0 \rangle = \langle v_i, v_j \rangle + \beta^2 - \beta \langle v_i + v_j, v_0 \rangle$$
  
$$\leq \langle v_i, v_j \rangle + \beta^2 - 4\epsilon \beta. \tag{12}$$

Altogether,

$$0 \leq \left\langle \sum_{i \in [L']} (v_i - \beta v_0), \sum_{i \in [L']} (v_i - \beta v_0) \right\rangle$$

$$= \sum_{i \in [L']} \left\langle v_i - \beta v_0, v_i - \beta v_0 \right\rangle +$$

$$\sum_{1 \leq i < j \leq L'} \left\langle v_i - \beta v_0, v_j - \beta v_0 \right\rangle$$

$$\leq L'(1 + \beta^2 - 4\epsilon\beta) +$$

$$2L'^2 \epsilon^2 + (L'^2 - L')(\beta^2 - 4\epsilon\beta),$$

where the last inequality is using (9), (11), and (12). Therefore, after reordering, we have  $L' \leq 1/(4\epsilon\beta - \beta^2 - 2\epsilon^2)$ , provided that the denominator is positive. Now we choose  $\beta := \epsilon$  to get  $L' \leq 1/\epsilon^2$ . Therefore, it suffices to choose  $L' > \max\{1/\epsilon^2, L\}$  to get the desired contradiction.

A sequence of results that we have seen so far can be combined to obtain list-decodable codes from RIP matrices. Namely, starting from a binary RIP matrix, we can apply Proposition 20, Lemma 24, and Lemma 29 in order and obtain the following:

**Lemma 30.** Suppose an  $n \times N$  matrix M with entries in  $\{-1/\sqrt{n}, +1/\sqrt{n}\}$  satisfies the RIP-2 of order L with constant  $\alpha$ . Let  $C \subseteq \mathbb{Z}_2^n$  be the binary code such that  $M = \operatorname{Sph}(C)$ . Then, there is a parameter  $\epsilon_0 = O(\sqrt{\alpha/L})$  such that for every  $\epsilon \geq \epsilon_0$ , C is  $(O(1/\epsilon^2), 1/2 - \epsilon)$ -list decodable.

Recall that the probabilistic method shows that RIP-2 matrices of order L exist with N columns and  $n=O(L\log(N/L))$  rows, and this is achieved with overwhelming probability by a random matrix (with  $\pm 1/\sqrt{n}$  entries). Using such a matrix in the above lemma, we obtain an  $(O(1/\epsilon^2), 1/2 - \epsilon)$ -list decodable code with rate  $R=\Omega(\epsilon^2)$ . It can be directly shown that this list-decoding trade-off is achieved by random codes with overwhelming probability, and the trade-off is essentially optimal (cf. [30]). However, explicit construction of optimal RIP-2 matrices and optimal binary list-decodable codes at radius  $1/2 - \epsilon$  are both challenging open problems. Therefore, Lemma 30 relates two important explicit construction problems; namely, it implies a reduction from Problem 31 to

Problem 32 below<sup>4</sup> (when the latter problem is restricted to binary real matrices).

**Problem 31.** Construct an explicit family of binary codes with block length n and rate  $R = \Omega(\epsilon^2)$  that are  $(O(1/\epsilon^2), 1/2 - \epsilon)$ -list decodable.

**Problem 32.** Construct an explicit family of RIP-2 matrices of order L with N columns and  $n = O(L \log(N/L))$  rows.

In Section III we showed how to obtain explicit RIP-2 matrices from spherical embedding of codes on the Gilbert-Varshamov bound constructed by Porat and Rothschild [17]. This construction achieves  $n=O(L^2\log N)$ , which achieves the best known explicit bound for matrices with  $\pm 1/\sqrt{n}$  entries<sup>5</sup>. Observe that the dependence on L is sub-optimal by a factor two in the exponent. As for binary list-decodable codes at radius close to 1/2 (and small list-size), Guruswami et al. construct explicit  $(O(1/\epsilon^2), 1/2 - \epsilon)$ -list decodable codes of rate  $R = \Omega(\epsilon^4)$  [30]. Again, the exponent of  $\epsilon$  in the rate is sub-optimal by a factor two.

A natural question is whether the reduction offered by Lemma 30 holds in the reverse direction as well; namely,

**Question 33.** Let  $C \subseteq \mathbb{Z}_2^n$  be such that, for some integer L and every  $1 < L' \le L$ , the code C is  $(L', 1/2 - O(\sqrt{\alpha/L'}))$ -list decodable. Does Sph(C) satisfy RIP-2 of order  $\Omega(L)$  with constant  $O(\alpha)$ ?

From Lemmas 23 and 21 we know that in order to answer the above question in affirmative, it suffices to show a converse to Lemma 29. A weak converse, not strong enough for this purpose, is shown below.

**Lemma 34.** Suppose that a code  $C \subseteq \mathbb{Z}_2^n$  is  $(L, 1/2 - \epsilon)$ -list decodable. Then, for  $L' := L/\epsilon$ , the L'-wise distance of the code  $C \subseteq \mathbb{Z}_2^n$  is at least  $1/2 - 2\epsilon$ .

*Proof:* The proof is, in essence, a straightforward averaging argument. Suppose, for the sake of contradiction, that there is a set of L' codewords whose average distance is less than  $1/2 - 2\epsilon$ . Denote the spherical encodings of these codewords by  $c_1, \ldots, c_{L'}$ , each in  $\{-1, +1\}^n$ . From the definition of L'-wise distance (Definition 18), we have

$$\sum_{1 \le i < j < L'} \langle c_i, c_j \rangle > 4\epsilon \binom{L'}{2} n > 2\epsilon L'^2 n = 2L^2 n/\epsilon. \quad (13)$$

Now, define  $v:=\sum_{i=1}^{L'}c_i/L'$  and note that this is a real vector in  $[-1,+1]^n$ . Suppose  $v=(v_1,\ldots,v_n)$  and randomly pick a vector  $\bar{v}=(\bar{v}_1,\ldots,\bar{v}_n)\in\{-1,+1\}^n$  with independent coordinates such that  $\mathbb{E}[\bar{v}_i]=v_i$ . This is possible since each  $v_i$  is in [-1,+1].

Note that, by linearity of expectation, for every i we have  $\mathbb{E}[\langle \bar{v}, c_i \rangle] = \langle v, c_i \rangle$ . Again, using linearity of expectation,

$$\mathbb{E}\left[\sum_{i=1}^{L'} \langle \bar{v}, c_i \rangle\right] = \sum_{i=1}^{L'} \langle v, c_i \rangle = \langle v, \sum_{i=1}^{L'} c_i \rangle$$

$$= \frac{1}{L'} \langle \sum_{i=1}^{L'} c_i, \sum_{i=1}^{L'} c_i \rangle$$

$$= \frac{1}{L'} (L' + 2 \sum_{1 \le i < j \le L'} \langle c_i, c_j \rangle)$$

$$\stackrel{(13)}{>} 1 + 4Ln > 4Ln. \tag{14}$$

Since there is a choice of the randomness that preserves expectation, we can ensure that there is a deterministic choice of  $\bar{v} \in \{-1, +1\}^n$  that satisfies (14). In the sequel, fix such a  $\bar{v}$ . We thus have

$$\sum_{i=1}^{L} \langle \bar{v}, c_i \rangle > 4Ln. \tag{15}$$

Now, (14) implies that there must be a set S of more than L vectors in  $\{c_1,\ldots,c_{L'}\}$  such that for every  $c\in S$ , the inequality  $\langle v,c\rangle>2\epsilon n$  holds, since if this were not the case, we would have

$$\sum_{i=1}^{L} \langle \bar{v}, c_i \rangle \le L'(2\epsilon n) + Ln = 3Ln,$$

contradicting (15). We conclude that the set of codewords corresponding to the spherical encodings in S are all  $(1/2-\epsilon)$ -close in Hamming distance to the binary vector represented by  $\bar{v}$ . This contradicts the assumption that C is  $(L, 1/2 - \epsilon)$ -list decodable and completes the proof.

#### VII. CONCLUSION

The reductions between coding-theoretic objects such as codes with large distance, incoherent spherical codes, combinatorial designs and the like are not only interesting for constructions, but also they relate the known bounds on the parameters achievable by one to another. For example, due to the reduction from binary codes to spherical codes, any improved lower bound on the coherence of spherical codes results in an improved upper bound on the rates achievable by small-biased codes. Thus, it is interesting to explore further connections of this type. For example, whether there is a reduction from disjunct matrices to designs, designs to codes, etc. Moreover, an affirmative answer to Question 33 would imply that the seemingly unrelated problems of finding explicit RIP-2 matrices (with  $\pm 1/\sqrt{n}$  entries) and explicit binary list-decodable codes<sup>6</sup> at radius close to 1/2 are essentially equivalent. In particular, optimal RIP-2 matrices would imply optimal binary list-decodable codes and vice versa. One can also ask similar questions about non-binary codes, which

<sup>&</sup>lt;sup>4</sup> We remark that, for the reduction to yield explicit list-decodable codes, an explicit algorithm that computes the RIP matrix in polynomial time in the size of the matrix would not necessarily suffice. One needs the more stringent explicitness that requires each individual entry of the matrix to be computable in time poly(n).

<sup>&</sup>lt;sup>5</sup>Bourgain et al. [23] explicitly obtain a better-than-quadratic dependence on L for an interesting range of parameters. However, entries of their matrices are powers of the primitive complex pth root of unity for a large prime p.

<sup>&</sup>lt;sup>6</sup>Note that there is no requirement on the existence of an efficient list-decoder for the code. Only the encoding function needs to be efficient.

might be easier to construct, or consider related variations of the *L*-wise distance<sup>7</sup>.

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 $^{7}$ One possibility is to look at the spherical embedding of the code and work with average inner products of pairs within all collections of L codewords, rather than the average L-wise distance as in Definition 17.

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